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A Complete Solution to the Existence and Nonexistence of Knut Vik Designs and Orthogonal Knut Vik Designs

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Hedayat and Federer (*Ann. of Statist.* **3** (1975), 445–447) proved that Knut Vik designs do not exist for all even orders. They gave a simple algorithm for the construction of such designs for all other orders, except when the order of the design is divisible by 3. The existence of Knut Vik designs of orders divisible by 3 was left unsolved by these authors. It is shown here that Knut Vik designs do not also exist for all orders divisible by 3. An easy algorithm based on a result of Euler is provided for the construction of orthogonal Knut Vik designs for all orders not divisible by 2 or 3. Therefore, we can say that Knut Vik designs and orthogonal Knut Vik designs of order n exist if and only if n is not divisible by 2 or 3. The results are based on the concepts of a super diagonal and parallel super diagonals in an $n \times n$ array, which have been introduced and studied for the first time here. Other relevant results are also given.

1. INTRODUCTION

Let A be an $n \times n$ array. Index its rows and columns by $0, 1, \ldots, n-1$. By the jth right diagonal of A we mean the following n cells of A:

$$(i, j+1); \qquad i = 0, 1, 2, \ldots, n-1; \qquad (\mathrm{mod}\, n). \tag{1.1}$$

Except for $j = 0$, all other right diagonals are broken. Also, we define the jth left diagonal of A to be the following n cells of A:

$$(i, j-i-1); \qquad i = 0, 1, 2, \ldots, n-1; \qquad (\mathrm{mod}\, n). \tag{1.2}$$

Again, except for $j = 0$ all other left diagonals are broken. Some writers refer to these diagonals as broken diagonals when $j \neq 0$. Let Σ be a set of n distinct elements (treatments). If we can fill the cells of A by the elements of Σ in such a way that each row, column, right diagonal, and left diagonal of A contains all the elements of Σ, then we say the resulting structure, which

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we denote by K, is a Knut Vik design of order n. Note that a Knut Vik design is necessarily a Latin square design. The statistical applications of such designs can be found in Hedayat and Federer [3] and the list of references given there. An example of such a design when $n = 5$ and $\Sigma = \{0, 1, 2, 3, 4\}$ is given below:

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{array}$$

Now a question of interest is this: Can we construct a Knut Vik design for every n? Unfortunately the answer is no. As Hedayat and Federer [3] have shown, these designs cannot be constructed if n is even. A simple algorithm for the construction of these designs is given by Hedayat and Federer [3] if n is not a multiple of 2 or 3. The existence of these designs for orders divisible by 3 was left unsolved by these authors. It is shown here that, unfortunately, Knut Vik designs do not also exist for all orders divisible by 3 (Section 2). Therefore, we can say a Knut Vik design of order n exists if and only if n is not dividible by 2 or 3 .To prove this result, and others, we have first introduced the concepts of a super diagonal and parallel super diagonals for A (Section 2). It is shown that A can be filled with the elements of Σ to form a Knut Vik design of order n if and only if we can decompose A into n parallel super diagonals (Section 2). Then it is shown that if n is divisible by 2 or 3 then A does not have even a single super diagonal (Section 2). Thus no Knut Vik design of order n can be constructed if n is divisible by 2 or 3. To prove our results we have utilized some number theoretic arguments of Euler [1]. The idea of orthogonal Knut Vik designs is introduced and it is shown that orthogonal Knut Vik designs of order n can be constructed if n is not divisible by 2 or 3 (Section 4). Simple algorithms for the construction of Knut Vik designs for all admissible n are also given (Section 3). The problem of statistical optimality of Knut Vik designs and orthogonal Knut Vik designs is currently under our investigation and the corresponding results will be reported elsewhere.

2. Super Diagonals

The concepts of super diagonal and parallel super diagonals are very useful for investigating the existence and nonexistence of Knut Vik designs. These concepts are introduced formally below.

DEFINITION 2.1. A collection of n cells in A is said to be a super diagonal in A if each row, column, left, and right diagonal of A has a cell in this collection.

EXAMPLE 2.1. The corssed cells in the following array are super diagonals:

	0	1	2	3	4
0	×		·		
1			×		·
2		·			×
3		×		·	
4	·			×	

We now characterize the n cells of a super diagonal. Let $S = \{(x_i, y_i), i = 0, 1, 2, \ldots, n-1\}$ be a super diagonal in A; then we have

LEMMA 2.1. *The necessary and sufficient conditions for the set of cells S to be a super diagonal are*

(i) $\{x_i: i = 1, 2, \ldots, n\} = \Sigma$;

(ii) $\{y_i: i = 1, 2, \ldots, n\} = \Sigma$;

(iii) $\{y_i - x_i (\mathrm{mod}\, n): i = 1, 2, \ldots, n\} = \Sigma$;

(iv) $\{y_i + x_i (\mathrm{mod}\, n): i = 1, 2, \ldots, n\} = \Sigma$.

Proof. The result follows at once by noting that the cells of the jth row are $\{(x_i, y_i): x_i = j\}$, the cells of the jth column are $\{(x_i, y_i): y_i = j\}$, the cells of the jth right diagonal are $\{(x_i, y_i): y_i - x_i = j, (\mathrm{mod}\, n)\}$, and the cells of the jth left diagonal are $\{(x_i, y_i): y_i + x_i = j - 1, \mathrm{mod}\, n\}$.

EXAMPLE 2.2. The super diagonal in Example 2.1 is

$$S = \{(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)\},$$

and the reader can check for himself that the row and column indices of S satisfy four conditions of Lemma 2.1.

LEMMA 2.2. *An $n \times n$ array has a super diagonal if and only if n is not divisible by* 2 *or* 3.

Proof. If $S = \{(x_i, y_i): i = 1, 2, \ldots, n\}$ is a super diagonal then, from Lemma 2.1,

$$\sum_{i=1}^{n-1} i = \sum_{i=0}^{n-1} (y_i - x_i) = \sum_{i=0}^{n-1} y_i - \sum_{i=0}^{n-1} x_i = 0 \qquad (\text{mod } n).$$

Hence,

$$n(n-1)/2 = 0 \qquad (\text{mod } n),$$

which is impossible if n is even.

Again from Lemma 2.1,

$$2 \sum_{i=0}^{n-1} x_i y_i = \sum_{i=0}^{n-1} x_i^2 + \sum_{i=0}^{n-1} y_i^2 - \sum_{i=0}^{n} (y_i - x_i)^2 = \sum_{i=0}^{n-1} i^2 \qquad (\text{mod } n),$$

$$4 \sum_{i=0}^{n-1} x_i y_i = \sum_{i=0}^{n-1} (y_i + x_i)^2 = \sum_{i=0}^{n-1} (y_i - x_i)^2 = 0 \qquad (\text{mod } n).$$

Hence

$$2 \sum_{i=0}^{n-1} i^2 = n(n-1)(2n-1)/3 = 0 \qquad (\text{mod } n),$$

which is impossible if n is divisible by 3. As we shall see shortly, for all other n's the array has a super diagonal.

We now introduce the concept of parallel super diagonals.

DEFINITION 2.2. Two super diagonals are said to be parallel if they have no cell in common.

In Example 2.1, the crossed cells and the cells identified by $\cdot$ are two parallel super diagonals.

THEOREM 2.1. *A Knut Vik design of order n exists if and only if there exists an $n \times n$ array with n parallel super diagonals.*

Proof. Let A be an $n \times n$ array with n parallel super diagonals. If we fill the cells of each super diagonal with a fixed element of Σ then the resulting structure will be a Knut Vik design, if no two super diagonals are filled with the same element. Conversely, if K is a Knut Vik design of order n, then each set of n cells of K with a fixed entry is a super diagonal. Clearly, for two different elements the corresponding super diagonals are parallel.

COROLLARY 2.1. *A Knut Vik design of order n exists if and only if n is not divisible by* 2 *or* 3.

This is a consequence of Lemma 2.2, Theorem 2.1, and the result of Hedayat and Federer, given by [3, Corollary 2.1].

Using different terminology, Euler [1] has shown that it is impossible to rearrange rows and columns of the multiplication table of the cyclic group of order $3k$ to form a Knut Vik design. But as we have shown above, such designs cannot be constructed, anyway.

If n is a prime, then Hedayat and Federer [3] have given a simple algorithm for the construction of a Knut Vik design of order n. If n is not a prime, then the Kronecker product method is suggested by Hedayat and Federer for the construction of a Knut Vik design. Here we want to point out another method of constructing Knut Vik designs for all composite n's not divisible by 2 or 3.

3. An Algorithm for the Construction of Knut Vik Designs

Following Euler [1] one can easily prove:

(i) If $n > 3$ is a prime, then

$$K = (k_{ij}) \qquad \text{with} \qquad k_{ij} = \lambda i + j \pmod n, \qquad \lambda \neq 0, 1, n-1.$$

is a Knut Vik design of order n.

(ii) If n is not divisible by 2 or 3 and

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t},$$

then $K = (k_{ij})$ with $k_{ij} = \lambda i + j \pmod n$ is a Knut Vik design if λ, $\lambda - 1$, $\lambda + 1$ are relatively prime to n. This means that for all r the following values are not allowed to be given to λ

$$rp_i, \qquad rp_i + 1, \qquad rp_i - 1, \qquad i = 1, 2, \ldots, t.$$

In this case we have

$$N = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_t^{\alpha_t - 1} (p_1 - 3)(p_2 - 3) \cdots (p_t - 3) \tag{3.1}$$

different choices for λ.

Example 3.1. Let $n = 35$. In this case N in (3.1) becomes $(5-3)(7-3) = 8$, and λ can take any value from the set

$$\{2, 3, 12, 17, 18, 23, 32, 33\}.$$

To use the method of Hedayat and Federer [3] for the purpose of constructing a Knut Vik design of order 35, one has to construct a Knut Vik design of order 5 and another of order 7 and then form the Kronecker product of these designs. The procedure outlined above is direct and simple. In a different contex, Hudson [4] has given a method for the construction of these designs. But it should be mentioned that Hudson's method is a special case of the above algorithm.

Note that Knut Vik designs constructed by the preceding algorithm are all cyclic. An unsolved and interesting problem in this area is to find methods for constructing noncyclic Knut Vik designs. In this regard, it is not difficult to show that all Knut Vik designs of order 5 are cyclic. It has been verified by a computer that Knut Vik designs for orders 7 and 11 are also cyclic. Noncyclic Knut Vik designs of order 13 exist. A related problem, namely, the number of nonisomorphic Knut Vik designs for a given order is currently under study by A. O. L. Atkin, L. Hay, and R. G. Larson at the Department of Mathematics, University of Illinois, Chicago.

4. Orthogonal Knut Vik Designs

Whatever properties orthogonal Latin squares may have will be enjoyed as well by orthogonal Knut Vik designs. For this reason, and the fact that Knut Vik designs have additional applications, we shall introduce and study the concept of orthogonal Knut Vik designs here.

Definition 4.1. If K_1 is a Knut Vik design of order n, and if K_2 is a Knut Vik design of order n, then we say K_1 is orthogonal to K_2 if K_1 and K_2 are orthogonal in the sense of Latin squares.

Example 4.1. Let $n = 5$; then the following two Knut Vik designs are orthogonal:

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{array} \qquad\qquad \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \end{array}$$

If $K_1, K_2, \ldots, K_t$ are t Knut Vik designs of order n such that K_i is orthogonal to K_j, $i \neq j$, then a problem of interest is to find the upper bound on t and investigate those cases where the upper bound can be reached. In this regard we have the following theorem.

THEOREM 4.1. *If $K_1, K_2, \ldots, K_t$ are t pairwise orthogonal Knut Vik designs of order n, then $t \leqslant n-3$.*

Proof. These t Knut Vik designs, together with

$$A = (a_{ij}), \qquad a_{ij} = i + j \pmod n,$$
$$B = (b_{ij}), \qquad b_{ij} = -i + j \pmod n,$$

form a set of $t+2$ pairwise orthogonal Latin squares of order n. But it is well known that $t+2 \leqslant n-1$, which implies that $t \leqslant n-3$.

In Section 3 we said that for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ and not divisible by 2 or 3, the following array is a Knut Vik design:

$$K = \lambda i + j \pmod n,$$

with λ being able to take N (see (3.1)) possible values. Let λ_1 and λ_2 be two such choices. Then it is not hard to prove that

$$K_1 = \lambda_1 i + j \pmod n,$$
$$K_2 = \lambda_2 i + j \pmod n,$$

are orthogonal if $\lambda_1 - \lambda_2$ and n are relatively prime. Therefore we have

THEOREM 4.2. *If n is a prime there are $n-3$ pairwise orthogonal Knut Vik designs. If n is not a prime and if n is not divisible by 2 or 3 then there is at least a pair of orthogonal Knut Vik designs.*

EXAMPLE 4.2. If $n = 35$, then $\lambda_1 = 2$ and $\lambda_2 = 18$ can be used to construct a pair of orthogonal Knut Vik designs of order 35.

One can also use the technique of Kronecker products for the construction of orthogonal Knut Vik designs.

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